The structure of a balanced axi-symmetric vortex

$$\text{gradient wind balance} \quad \frac{\partial p}{\partial r} = \rho \left(\frac{v^2}{r} + fv \right) \tag{1}$$

$$\text{hydrostatic balance} \quad \frac{\partial p}{\partial r} = -\rho g \tag{2}$$

Cross differentiating gives a form of the thermal wind

$$\begin{aligned} \frac{\partial^2 p}{\partial z \partial r} &= \rho \left(\frac{2v}{r} + f \right) \frac{\partial v}{\partial z} + \left(\frac{v^2}{r} + fv \right) \frac{\partial \rho}{\partial z} \\ &= \frac{\partial^2 p}{\partial r \partial z} = -g \frac{\partial \rho}{\partial r} \end{aligned}$$

Simplification gives

$$g \frac{\partial}{\partial r} \ln \rho + \left(\frac{v^2}{r} + fv \right) \frac{\partial}{\partial z} \ \ln \ \rho = - \left(\frac{\partial v}{r} + f \right) \frac{\partial v}{\partial z}. \tag{3}$$

This is a linear first-order differential equation for the density distribution as a function of r and z when the tangential wind distribution, $v(r, z)$, is prescribed.

The characteristics of the equation are given by

$$\frac{dz}{dr} = \frac{\left(\frac{v^2}{r} + fv \right)}{g} \tag{4}$$

Along these characteristics,

$$\frac{d}{dr} \ln \left(\frac{\rho}{\rho_0} \right) = - \frac{\left(\frac{2v}{r} + f \right)}{g} \frac{\partial v}{\partial z} \tag{5}$$

The characteristics are simply isobaric surfaces, since a small displacement (dr, dz) along such a surface is such that

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial z} dz = 0 \tag{6}$$

Using (1) and (2) to eliminate $\partial p / \partial r$ and $\partial p / \partial z$ gives (4).

Now, given an environmental sounding characterized by $p_0(z)$ and $\rho_0(z)$, we can integrate the pair of ordinary differential equations (4) and (5) radially inwards to find the height of the isobaric surface and the variation of density along it.

Note that for a barotropic vortex, $\partial v / \partial z = 0$, and from (5) it follows that $\rho =$ constant along an isobaric surface, i.e. $\rho = \rho(p)$.

Equation (5) shows that for a cyclonic vortex ($v > 0$) with $dv/dz > 0$, $log(\rho/\rho_0)$ and hence ρ decreases with decreasing radius along the isobaric surface so that the temperature $T(r)$ increases and the vortex is warm cored. Conversely, if $dv/dz > 0$, the vortex is cold cored.

The structure of a balanced geostrophic flow

A special case of (1) is when there is geostrophic balance (i.e. the centrifugal term is not present). Consider a geostrophic flow $u(y, z)$ in the x-direction. Then

$$\text{geostrophic balance} \quad \frac{\partial p}{\partial y} = -\rho f u \tag{7}$$

$$\text{hydrostatic balance} \quad \frac{\partial p}{\partial z} = -\rho g \tag{8}$$

Cross-differentiation now gives

$$\frac{\partial^2 p}{\partial z \partial y} = -\rho f \frac{\partial y}{\partial z} - f u \frac{\partial \rho}{\partial z} = \frac{\partial^2 p}{\partial y \partial z} = -g \frac{\partial \rho}{\partial y},$$

or

$$g \frac{\partial \rho}{\partial y} - f u \frac{\partial \rho}{\partial z} = \rho f \frac{\partial y}{\partial z}, \tag{9}$$

Which is analogous to (3). Now the characteristics satisfy

$$\frac{dz}{dy} = -\frac{fu}{g} \tag{10}$$

and along these

$$\frac{d}{dy} \ln \left(\frac{\rho}{\rho_0} \right) = -f \frac{\partial u}{\partial z} \tag{11}$$

In this case, a displacement (dy, dz) along an isobaric surface gives

$$dp = \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz = 0$$

so that dz/dy is simply given by (10). Again the isobaric surfaces are the characteristics of (9) and (11) determines the density variation along a characteristic.

If the flow is barotropic, $\partial u/\partial z = 0$ and again $\rho = \rho(p)$.

If $\partial u/\partial z > 0$, (11) tells us that $\ln \rho$ and hence ρ increase in the positive y-direction, whereupon T decreases in this direction at constant z.

The nice thing about these calculations is that we make no assumption about density (i.e. no Boussinesq or anelastic approximation).